Ordering near percolation threshold in models of interacting bosons with quenched dilution
Texas A&M University, Feb. 2006

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How does strong correlation (antiferromagnetism, superconductivity) spread through an impure system?

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Thanks:
D.-H. Lee (Berkeley)
M. Greven (Stanford)

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Pervasive Inhomogeneity(I): Quantum Hall Effect

A man running by the system at speed \( v = \frac{E}{B} \) measures no electric field, and hence no current.

\[ E' = E + v \times B = 0 \]

Bottom line: Disorder not completely negligible.
Pervasive inhomogeneity (II): Colossal magneto-resistance


The magnetic regions (yellow) are surrounded by non-magnetic regions (red).

Cartoon picture of CMR: magnetic field introduces magnetic patches. With enough
Pervasive Inhomogeneity (III):

Scanning tunneling microscopy of a high-temperature superconductor ($T_c = 73K$). Gap in electronic density of states varies by 4% (Blue and yellow both have gaps)

Blue regions are not really superconducting:

They have no single-particle coherence peaks above the gap, unlike yellow.

They never have mid-gap, localized states around impurities.

Optical lattice of interacting bosons

The tunnel coupling $J$ between sites acts to correlate their phases, and hence reduce phase fluctuations on each bond.

There is also a contact interaction $U$. The contact interaction energy acts to reduce number fluctuations on each site.

There is competition between a Mott insulating phase (charge eigenstate), and a superfluid phase (phase eigenstate).
Diluted optical lattice: Simple model of disorder

\[ H = \frac{U}{2} \sum_i n_i^2 - J \sum_{\langle ij \rangle} \alpha_{ij} \cos(\phi_i - \phi_j). \]

\[ \phi \text{ generates boson addition:} \]

\[ [n_i, \exp i \phi_j] = \delta_{ij} \exp i \phi_i. \]

\[ \alpha_{ij} = 0, \text{ with probability } z \]
\[ = 1, \text{ with probability } 1-z \]
Diluted optical lattice: Self-similar structure

$z = 1/8$

Black sites can talk

$z = 0.407$

$z = 7/8$

Black sites can’t talk

Self-similar structure with $d = 91/48$

“between one and two-dimensional”

Reminder:
With probability $z$, site is removed
With probability $(1-z)$, site remains.
Debye-Waller factor...

Suppose $U/J << 1$, $z \sim z_c$, $T \sim 0$.

In time-of-flight, we expect Bragg peaks.

We compute the Debye-Waller factor.

\[
\langle \exp(i\phi) \rangle = \exp\left(-\frac{1}{2} \langle \phi_i^2 \rangle \right)
\]

\[
\langle \phi_i^2 \rangle = \int d\omega \left[ \left( \frac{\omega^2}{U} + JT \right)^{-1} \right]_{ii}
\]

\[
= \sqrt{\frac{U}{J}} [T^{-\frac{1}{2}}]_{ii}
\]

Hohenberg (1969): For finite $U$, DW factor diverges in true 1-d lattice (e.g., graphene).
Debye-Waller factor… is finite!

\[ <\phi^2>_{Perc} = \int_0^\infty d\lambda \frac{\rho(\lambda)}{\sqrt{\lambda}} \]

\[ \rho(\lambda) \sim \lambda^{\frac{d_s}{2} - 1} \]

Hohenberg case: \( ds=1.00 \), DW factor diverges for non-zero \( U \).
Percolation case: \( ds=1.32 \), (Alexander-Orbach), DW finite at small \( U \).
But...many weak links

A “weak link” is a bond that, when cut, separates point A from point B. There is a diverging number of red bonds between A and B!

For points separated by $r$, $L_R \sim r^{3/4}$. Individual quasi-2D blobs seem to order, but when does this order persist through the
Quantum antiferromagnetism in low dimensions

\[ H = J \sum_{\langle ij \rangle} S_i \cdot S_j \]

\( J > 0, \; S_i \) are spin-1/2 operators:

\[ [S_i^\alpha, S_j^\beta] = i \delta_{ij} \epsilon_{\alpha\beta\gamma} S_i^\gamma \]

has an ordered ground state, on the square lattice \((d=2)\)...

\[ \langle S_i \cdot S_j \rangle \rightarrow 0.307 \cdot (-1)^{i+j} \]

a disordered ground state, on a chain \((d=1)\)...

\[ \langle S_i \cdot S_j \rangle \rightarrow \frac{(-1)^{(i+j)}}{|r_{ij}|} \]

No “frustration”, not a spin glass: unique classical and quantum ground-state.
Quantum magnetism: Ferro and Antiferro

Classically, there is no fundamental difference between ferromagnets and antiferromagnets on a bipartite lattice. Can map between the two problems by flipping spins on one sublattice.

What about quantum mechanics?

Even at the level of two coupled Heisenberg spins, quantum mechanics breaks the symmetry between ferromagnet and antiferromagnet.
Order through disorder: the superconductor or quantum antiferromagnet near $p_c$

At first glance it seems like combining disordered one-dimensional links with ordered two-dimensional blobs should simply give a disordered phase.

Both the spin-half AF and the superconducting model below the KT transition have long-range order on the blobs, but only algebraic order on the links. Above the KT transition, or for spin-one, there is no order (exponential decay of correlations) on the links.

Quantum systems can behave quite bizarrely at zero temperature, so the ground state could still have LRO, as suggested by the following example.
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Both the spin-half AF and the superconducting model below the KT transition have **long-range order** on the blobs, but only **algebraic order** on the links. Above the KT transition, or for spin-one, there is **no order** (**exponential decay of correlations**) on the links.

Quantum systems can behave quite bizarrely at zero temperature, so the ground state could still have LRO, as suggested by the following example.

If the spins A and B at the ends are attached weakly ($J' \ll J$), then even though the next-to-last spins are only weakly correlated in the ground state (assumed nondegenerate), spins A and B form a perfect singlet. Exact diagonalization on spin chains of length up to $N=20$ confirms the argument.
Solvable case: slow, semi-classical blobs

This case can be solved because the blobs just give boundary conditions on the 1D links.

\[ P(J)dJ \sim \int dL e^{-aL} \delta(J - J_0 e^{-L/\zeta}) \]

The transition between stiff and floppy phases is driven by competition between the exponentially low energy to fluctuate across a link, and the exponential rarity of such links.
Stiff and floppy phases

Spin stiffness: Energy cost to impose a twist in direction of order

At finite temperature, order disappears; stiffness determines how quick

\[ \lambda(T) \sim e^{2\pi\rho_s/T} \]

Stiff phase: Twist taken up entirely across blobs
Stiffness scales classically. (t, v universal)

Floppy phase: Twist taken up entirely across links.
Stiffness scales anomalously (t NON-universal).

\[ \rho_{s}(R, p = p_c) \sim R^{-t/\nu} \]

Bottom line: Slow blob approximation says s=1/2 antiferromagnet orders
Quantum destruction of stiffness

The model problem of ordered regions connected by links of random length shows that the real problem is “close to” a stiffness transition in a certain parameter space. The stiffness is reduced by both the geometry of percolation and quantum fluctuations in quasi-1D regions.

One consequence is that at low temperature the correlation length is much shorter than in the undiluted 2D system. The classical geometric critical point $p_c$ looks like a QPT:

\[
\lambda \sim T^{-\nu/t} \quad \text{and} \quad \lambda \sim (p-p_c)^{-\nu}
\]

\[
\lambda \sim \xi e^{2\pi c(p-p_c)t/T}
\]

\[
p = 0 \quad (2D) \quad p = p_c
\]
Slow Blob Approximation

- Slow-blob $\sim$ Born-Oppenheimer
- Ink $\sim$ electron
- Blob $\sim$ ion

\[ P(j, J, U) \propto \frac{j}{k_2}^{\alpha \zeta - 1} \]

\( \zeta \) diverges as \( U/J \) smaller.
\( \alpha \) is independent of \( U/J \).

\[ R(u, J, U) \propto \left( \frac{u}{U} \right)^\alpha \]

\( \alpha \) is negative (Stanley PRL 1984)

a. Treat blobs as parameters of link $\mathcal{L}$
b. Use result as a potential for blob dynamics
Real-space regrouping: Random rotor chain

\[ H = \frac{1}{2} \sum_{j} U_j \left( -i \frac{\partial}{\partial \varphi_j} \right)^2 - \sum_{j} J_j \cos(\varphi_{j+1} - \varphi_j). \]

Strong disorder implies a broad distribution of couplings and may lead to new fixed points. We implement a real-space renormalization group to study strong disorder.

\[ \tilde{J}_{i-1,i+1} = J_{i-1} J_i / \Omega. \]

\[ \tilde{C}_i = C_i + C_{i+1}. \]
Novel Finite Disorder Fixed Points

\[ P(U) dU \sim e^{-f_0/U} \]

\[ P(J) dJ \sim J^{1+g_0} \]


Bottom Line: Stiff and floppy phases survive beyond slow-blob approximation.

\[ P(U) dU \sim U^{-1+f_0} \]

\[ P(J) dJ \sim J^{-1+g_0} \]

Static and mobile holes in a cuprate antiferromagnet

LaCu\(_p\)(Zn,Mg)\(_{1-p}\)O\(_4\)

Dilution removes sites from lattice; doping only removes the electrons.

Recent fabrication techniques allow dilution down to \( p \sim 55\% \).

Large nearest-neighbor exchange: \( J \sim 1540\)K; \( J_{\text{NNN}}/J \sim 8\% \).
Experimental result of O. P. Vajk et al., Science (2002), for M(z).
Quantum Monte Carlo simulations of dilute quantum magnetism.

Magnetization versus dilution.

Quantum fluctuation versus dilution.

A. Sandvik PRB 2002.
Summary of results

1. Percolation transitions in both superconductors and antiferromagnets near zero temperature are governed by a competition between one- and two-dimensional physics.
2. Order through fast link: Exact diagonalization work suggests order despite fast fluctuations.

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1. ```Stiff” and ```floppy” phases: We computed the spin stiffness by approximating blobs as completely static. Argued that s=1/2 AF is “stiff.”
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1. RG argument: Blob-sizes increase under re-grouping. Blob fluctuations seem “irrelevant.” Supports picture that quasi-2d physics wins and order spreads through the cluster.

1. Future directions:
   a) Extend RG analysis to O(3) rotor and antiferromagnet.
   b) Problems with mobile electrons: kondo lattice, two-impurity kondo, …
Spin stiffness of site diluted $L \times L$ systems multiplied by $L^{(t/\nu)}$. The curve is a quadratic polynomial in $L^{-(d/2)}$. (A. Sandivk PRB 66, 024418 (2002).

Infinite-size extrapolated spin stiffness vs dilution fraction (circles). Slow approach to classical form near threshold (grey).