Cold Atoms
and
Out of Equilibrium Quantum Dynamics

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Two examples of complexity.

Single neuron – relatively easy to characterize.

~ $10^{10}$ neurons ???

NAND gate

Computers are complicated, but we understand them

Is “more” fundamentally different or just more complicated?
From single particle physics to many particle physics.

**Classical mechanics:** Need to solve Newton’s equation.

Single particle

\[ m \ddot{x} = f(x, t) \implies x(t + \delta t) = F(x(t), \dot{x}(t), t) \]

Many particles

\[ m_i \ddot{x}_i = f(x_1, \ldots, x_n, t) \implies \]
\[ x_i(t + \delta t) = F_i(x_1(t), \ldots, x_n, \dot{x}_1(t), \ldots \dot{x}_n(t), t) \]

Instead of one differential equation need to solve \( n \) differential equations, not a big deal!? With modern computers we can simulate thousands or even millions of particles.
Use specific numbers: $M=200$, $n=100$.

Fermions: $N_H = \frac{M!}{(M-n)!n!} \Rightarrow N_H \approx 10^{59}$

Bosons: $N_H = \frac{(M+n-1)!}{(M-1)!n!} \Rightarrow N_H \approx 10^{81}$

A computer built from all known particles in universe is not capable to exactly simulate even such a small system.
Other issues.

Typical level spacing for our system: \( \Delta E \sim \frac{1}{N_H} \sim 10^{-80} - 10^{-50} \text{ eV} \)

Gravitation field of the moon on electron: \( \frac{GMm_e}{R} \sim 10^{-6} \text{ eV} \)

Typical time scales needed to resolve these energy levels (e.g. to prepare the system in the pure state):

\[ \Delta t \sim \frac{\hbar}{\Delta E} \sim 10^{40} - 10^{70} \text{ s} \]

Required accuracy of the theory (knowledge of laws of nature) \( \sim 10^{-80}-10^{-50} \).

Many-body physics is fundamentally different from single particle physics. It *can not* be derived purely from microscopic description.
Experiments: best simulators are real systems.

Problem with solid state (liquid) systems – Hamiltonians are too complex

Cold atoms:
(controlled and tunable Hamiltonians, isolation from environment)

1. Equilibrium thermodynamics:
\[ Z = \text{Tr} \ e^{-\beta H} \]
Quantum simulations of equilibrium condensed matter systems

Experimental examples:
Optical Lattices:


OL are tunable (in real time): from weak modulations to tight binding regime.
Superfluid-insulator quantum phase transition in interacting bosons (from particles to waves).
Repulsive Bose gas.

\[ H = -\frac{\hbar^2}{2m} \sum_j \frac{\partial^2}{\partial x_j^2} + g \sum_{i<j} \delta(x_i - x_j) \]

\[ g \approx \frac{4\hbar^2 a_{3D}}{m a_{1}^2} \left( 1 - 1.46 \frac{a_{3D}}{a_{1}} \right)^{-1} \]

Lieb-Liniger, complete solution 1963.

\[ \gamma << 1 \]

\[ \gamma \sim 1 \]

\[ \gamma >> 1 \]

M. Olshanii, 1998

interaction parameter

\[ \gamma = \frac{E_{\text{int}}}{E_{\text{kin}}} \sim \frac{gn}{\hbar^2 n^2 / m} = \frac{gm}{\hbar^2 n} \]

Experiments:

T. Kinoshita, T. Wenger, D. S. Weiss., Science 305, 1125, 2004

Energy vs. interaction strength: experiment and theory.

No adjustable parameters!

Kinoshita et. Al., Science 305, 1125, 2004
Local pair correlations.

Kinoshita et. Al., Science 305, 1125, 2004
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1. Equilibrium thermodynamics:

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Quantum simulations of equilibrium condensed matter systems

2. Quantum dynamics:

\[ \rho(t) = e^{-iHt} \rho(t) e^{iHt} \]

Coherent and incoherent dynamics, integrability, quantum chaos, …
STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM

A one-dimensional dynamical system of 64 particles with forces between neighbors containing nonlinear terms has been studied on the Los Alamos computer MANIAC I. The nonlinear terms considered are quadratic, cubic, and broken linear types. The results are analyzed into Fourier components and plotted as a function of time.

The results show very little, if any, tendency toward equipartition of energy among the degrees of freedom.

\[ x_i' = (x_{i+1} + x_{i-1} - 2x_i) + \alpha [(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2] \]
\[ (i = 1, 2, \cdots, 64), \]
In the continuum this system is equivalent to an integrable KdV equation. The solution splits into non-thermalizing solitons Kruskal and Zabusky (1965).
Quantum Newton Cradle.
(collisions in 1D interacting Bose gas – Lieb-Liniger model)


No thermalization in 1D.
Fast thermalization in 3D.

Quantum analogue of the Fermi-Pasta-Ulam problem.
Cold atoms:
(controlled and tunable Hamiltonians, isolation from environment)

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   \[ Z = \text{Tr} e^{-\beta H} \]  
   Quantum simulations of equilibrium condensed matter systems

2. Quantum dynamics:
   \[ \varrho(t) = e^{-iHt} \varrho e^{iHt} \]  
   Coherent and incoherent dynamics, integrability, quantum chaos, …

3. = 1+2 Nonequilibrium thermodynamics?
   E.g. second law of thermodynamics.
Adiabatic process.

\[ x = x(t), \quad x(t_0) = x_0, \quad x(t_1) = x_f \]

Assume no first order phase transitions.

\[ x(t) = x_0 + \delta t, \quad t_1 = \frac{x_f - x_0}{\delta} \]

Adiabatic theorem: \( \delta \to 0 \Rightarrow S_B \approx S_A \Rightarrow \delta Q = 0 \)

“Proof”: \[ S_B = S_B(\delta) \]

then

\[ S_B(\delta) \approx S_A + \alpha \delta + \beta \delta^2 \]

\[ S_B(\delta) \approx S_A + \beta \delta^2 \]
Adiabatic theorem for integrable systems.

Density of excitations

\[ n_{\text{exc}} (\delta) = n_A + \tilde{\beta} \delta^2 \]

Energy density (good both for integrable and nonintegrable systems):

\[ E_B (\delta) = E_B (0) + \beta' \delta^2 \]

\( E_B(0) \) is the energy of the state adiabatically connected to the state A.

For the cyclic process in isolated system this statement implies no work done at small \( \delta \).
Adiabatic theorem in quantum mechanics

Landau Zener process:

\[ \hat{H} = \delta t \cdot \hat{\sigma}_z + \Delta \hat{\sigma}_x \]

\[ \rho_\uparrow(\infty) = e^{-\frac{\pi \Delta^2}{\delta \hbar}} \]

In the limit \( \delta \to 0 \) transitions between different energy levels are suppressed.

This, for example, implies reversibility (no work done) in a cyclic process.
**Adiabatic theorem in QM suggests adiabatic theorem in thermodynamics:**

1. Transitions are unavoidable in large gapless systems.
2. Phase space available for these transitions decreases with $\delta$. Hence expect

$$\delta \to 0 \implies \Delta n_{\text{ex}}(\delta) = n_{\text{ex}}(0) + \beta \delta^2$$

$$E_{\beta}(\delta) = E_{\beta}(0) + \tilde{\beta} \delta^2$$

Is there anything wrong with this picture?

Hint: low dimensions. Similar to Landau expansion in the order parameter.
More specific reason.

Equilibrium: high density of low-energy states $\rightarrow$

$$\rho(\varepsilon) = \int \frac{d^d k}{(2\pi)^d} \delta(\varepsilon - \varepsilon_k) \propto \varepsilon^{d/z-1}, \ \varepsilon_k \propto k^z$$

- strong quantum or thermal fluctuations,
- destruction of the long-range order,
- breakdown of mean-field descriptions,

Dynamics $\rightarrow$
population of the low-energy states due to finite rate $\rightarrow$

*breakdown of the adiabatic approximation.*
This talk: three regimes of response to the slow ramp:

A. Mean field (analytic) – high dimensions:

$$E_\beta (\delta) - E_\beta (0) \approx \beta \delta^2$$

B. Non-analytic – low dimensions

$$E_\beta (\delta) - E_\beta (0) \approx \beta |\delta|^r \quad r < 2$$

C. Non-adiabatic – lower dimensions

$$E_\beta (\delta) - E_\beta (0) \approx \beta |\delta|^r \sqrt{L^2} \quad r < 2 \quad L > 0$$
Example: crossing a QCP.

\[ \lambda = \delta t, \quad \delta \to 0 \]

Gap vanishes at the transition. No true adiabatic limit!

How does the number of excitations scale with \( \delta \)?

\[
\frac{d\Delta}{dt} \sim \Delta^2, \quad \Delta \sim (\delta t)^{2d} \implies \Delta \sim \delta^{\frac{2d}{2d+1}}
\]

\( n_{\text{ex}} \sim (\Delta)^{d/2} \sim \delta^{d/(2d+1)} \) — regime B for \( d < \frac{2d+1}{2} \)

A.P. 2003
Transverse field Ising model.

\[ \mathcal{H}_I = - \sum_j g \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \]

Phase transition at \( g=1 \).

\[ g \rightarrow 0 \Rightarrow \langle \sigma_i^z \sigma_j^z \rangle \rightarrow 1 \]

\[ g \rightarrow \infty \Rightarrow \langle \sigma_i^x \rangle \rightarrow 1 \]

Start at \( g>>1 \) and slowly ramp it to 0. Count the number of domain walls

Critical exponents: \( z=\nu=1 \Rightarrow d\nu/(z\nu+1)=1/2 \). \( n_{ex} \propto \sqrt{\delta} \)

A.P. 2003,
W. H. Zurek, U. Dorner, P. Zoller 2005,
J. Dziarmaga 2005
Possible breakdown of the Fermi-Golden rule (linear response) scaling due to bunching of bosonic excitations.

\[ \mathcal{H} = \sum_q \frac{\rho_s q^2}{2} |\phi_q|^2 + \frac{1}{2} \kappa_q \Pi_q^2 \]

\[ \kappa_q = \kappa + \lambda q^2 \]

\[ \omega_q = q \sqrt{\rho_s (\kappa + \lambda q^2)} \]

Hamiltonian of Goldstone modes: superfluids, phonons in solids, (anti)ferromagnets, …

In superfluids \( \kappa \) is determined by the interactions.

Imagine an experiment, where we start from a noninteracting superfluid and slowly turn on interaction.
Zero temperature regime:

\[ n_{\text{ex}} \propto |\delta|^{d/4} \delta^{d>4/3} \]
\[ n_{\text{ex}} \propto |\delta|^{1/3} L^{1/3} \quad d=1 \quad - \text{nonadiabatic regime C.} \]

Energy

\[ \mathcal{E} \propto \delta^{(d+1)/4}, \quad d < 7 \]

Assuming the system thermalizes at a fixed energy

\[ S \propto \delta^{d/4}, \quad T_f \propto \delta^{1/4} \]
Finite Temperatures

\[ \Delta E \propto T \delta^{1/3} L^{7/3-d} \]

\( d=1,2 \)

Non-adiabatic regime!

\[ \Delta S \propto \delta^{1/6} L^{2/3} \quad d=1; \quad \Delta S \propto (\delta L)^{2/9} \quad d=2; \]

\[ \Delta E \propto T \sqrt{\delta}, \Delta S \propto \sqrt{T} \sqrt{\delta} \quad d=3 \]

Artifact of the quadratic approximation or the real result?
Numerical verification (bosons on a lattice).

\[ \mathcal{H}_{bh} = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) + \frac{U(t)}{2} \sum_j a_j^\dagger a_j (a_j^\dagger a_j - 1), \]

\[ U(t) = U_0 \tanh(\delta t) \]

Nonintegrable model in all spatial dimensions, expect thermalization.

Use the fact that quantum fluctuations are weak in the SF phase and expand dynamics in the effective Planck's constant:

\[ \hbar \rightarrow \sqrt{U / n_0 J} \]
Quantum expansion of dynamics:

Leading order in $\hbar$: start from random initial conditions distributed according to the Wigner transform of the density matrix and propagate them classically (truncated Wigner approximation):

- Expectation value is substituted by the average over the initial conditions.
- Exact for harmonic theories!
- Not limited by low temperatures!
- Asymptotically exact at short times.

Subsequent orders: quantum scattering events (quantum jumps)
\[ T = 0.02 \]

\[ \Delta E \propto T L^{4/3} \delta^{1/3} \]
Thermalization at long times.

Correlation Functions

\[ \langle a_j^+ a_0 \rangle \]

- \( t=0 \)
- \( t=3.2/\delta \)
- \( t=12.8/\delta \)
- \( t=28.8/\delta \)
- \( t=51.2/\delta \)
- \( t=80/\delta \)

\[ \frac{L}{\pi} \sin \left( \frac{\pi j}{L} \right) \]
$2D, T=0.2$

\[ \Delta E \propto T L^{1/3} \delta^{1/3} \]
Conclusions.

Three generic regimes of a system response to a slow ramp:

A. Mean field (analytic):

\[ E_\beta(\delta) - E_\beta(0) \sim \beta \delta^2 \]

B. Non-analytic

\[ E_\beta(\delta) - E_\beta(0) \sim \beta |\delta|^{r} \quad r \leq 2 \]

C. Non-adiabatic

\[ E_\beta(\delta) - E_\beta(0) \sim \beta |\delta|^{r} L^2 \quad r \leq 2, \quad \varepsilon > 0 \]

Open questions: general fate of linear response at low dimensions, non-uniform perturbations,…
What happens if there is a current in the superfluid?

Drive a slowly moving superfluid towards MI.

possible experimental sequence:

~lattice potential

\[ \pi/2 \]

U/J

SF

MI

Unstable

Stable

p

p

SF

MI

U/J
Meanfield (Gutzwiller ansatzt) phase diagram

Is there current decay below the instability?
Role of fluctuations

Phase slip

Below the mean field transition superfluid current can decay via quantum tunneling or thermal decay.
1D System.

\[ \Gamma \propto \exp \left[ -7.1 \sqrt{\frac{JN}{U}} \left( \frac{\pi}{2} - p \right)^{5/2} \right] \]

7.1 – variational result

\[ \sqrt{\frac{JN}{U}} \]

semiclassical parameter (plays the role of \(1/\hbar))

Large \(N \sim 10^2 - 10^3\)

Fallani et al., 2004

C.D. Fertig et al., 2004
Higher dimensions.

Longitudinal stiffness is much smaller than the transverse.

\[ J_{\parallel} \rightarrow J \cos p, \]
\[ J_{\perp} = J \]

\[ r \propto 1 / \sqrt{\frac{\pi}{2} - p} \]

Need to excite many chains in order to create a phase slip.
Phase slip tunneling is more expensive in higher dimensions:

\[ S_d = \tilde{C}_d \sqrt{\frac{JN}{U}} \left( \frac{\pi}{2} - p \right)^{6-d} \]

\[ \Gamma_d \propto \exp(-S_d) \]

Stability phase diagram

\[ S_{1d} \approx 7.1 \sqrt{\frac{JN}{U}} \left( \frac{\pi}{2} - p \right)^{5/2} \]

\[ S_{2d} \approx 25 \sqrt{\frac{JN}{U}} \left( \frac{\pi}{2} - p \right)^2 \]

\[ S_{3d} \approx 93 \sqrt{\frac{JN}{U}} \left( \frac{\pi}{2} - p \right)^{3/2} \]

- \( S_d > 3 \) Stable
- \( 1 < S_d < 3 \) Crossover
- \( S_d < 1 \) Unstable
Current decay in the vicinity of the superfluid-insulator transition
Use the same steps as before to obtain the asymptotics:

\[ S_d = \frac{C}{\xi^{3-d}}\left(1 - \sqrt{3} p \xi\right)^{\frac{5}{2} - d}, \quad \Gamma \propto \exp(-S_d) \]

\[ S_1 \approx \frac{5.7}{\xi^2} \left(1 - \sqrt{3} p \xi\right)^{\frac{3}{2}} \]

\[ S_2 \approx \frac{3.2}{\xi} \left(1 - \sqrt{3} p \xi\right)^{\frac{1}{2}} \]

\[ S_3 \rightarrow 4.3 \]

Discontinuous change of the decay rate across the meanfield transition. Phase diagram is well defined in 3D!

Large broadening in one and two dimensions.
Detecting equilibrium SF-IN transition boundary in 3D.

Easy to detect nonequilibrium irreversible transition!!

At nonzero current the SF-IN transition is irreversible: no restoration of current and partial restoration of phase coherence in a cyclic ramp.
FIG. 3: Critical momentum for a condensate in a 3D lattice. The solid line shows the theoretical prediction for the superfluid region. The horizontal solid line is a fit to the data points in the MI phase. (Inset) Fit of critical momenta near the SF-MI phase transition.
Conclusions.

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C. Non-adiabatic \[ E_\beta (\delta) - E_\beta (0) \sim \beta |\delta|^{r/2} L^2 \quad r \leq 2, \quad 2 > 0 \]

Smooth connection between the classical dynamical instability and the quantum superfluid-insulator transition.

Depletion of the condensate. Reduction of the critical current. All spatial dimensions.

Quantum fluctuations

mean field

beyond mean field

Broadening of the mean field transition. Low dimensions