Allowed charge transfers between coherent conductors driven by a time-dependent scatterer

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AA and D. Ivanov, PRL 100, 086602 (2008)
Quantum Charge Transfer

- Charge pump: charge is transferred from left to right.
- The constriction - effective potential barrier.
- Gates + bias voltage - time-dependent scatterer.
- Open, wait, close.
- What is the quantum probability distribution of the transferred charge?
Motivation

\[ V(t) \]

Vanevic, Nazarov, Belzig, 2007

Charge transport - two independent processes
1. Binomial statistics due to the mean bias voltage
2. Symmetric contribution to noise

\[ p_q = p - q \]

\[ \bar{V} = 0 \]

\[ p_q - probability of transferring charge q to the right \]

Are there restrictions on charge transport for arbitrary scatterer?
**Full Counting Statistics**

\[
\chi(\lambda) = \left\langle e^{i\lambda(Q-Q_0)} \right\rangle = \sum_{q=-\infty}^{+\infty} p_q e^{i\lambda q}
\]

\[
p_q = \int_0^{2\pi} \frac{d\lambda}{2\pi} \chi(\lambda) e^{-i\lambda q}
\]

Levitov, Lesovik, 1993

- \( p_q \) - probability of transferring charge \( q \)

- \( \chi(\lambda) \) - FCS generating function

  compact and easy to calculate!

- \( \chi(0) = 1 \) - normalization

- \( \overline{\chi(\lambda)} = \chi(-\lambda) \) - reality

- \( \chi(\lambda + 2\pi) = \chi(\lambda) \) - periodicity (discreetness of charge)
FCS - cummulants

\[
\ln \chi(\lambda) = \sum_{k=1}^{\infty} \frac{\langle \delta q^k \rangle}{k!} \frac{(i\lambda)^k}{k!} \\
\chi(\lambda) = \sum_{q=-\infty}^{+\infty} p_q e^{i\lambda q}
\]

\[
\langle \delta q^k \rangle
\]
- irreducible correlators (cummulants)

exc. \[
\langle \delta q^1 \rangle = \bar{q}
\]
- mean (average transferred charge)

\[
\langle \delta q^2 \rangle = \bar{q}^2 - \bar{q}^2
\]
- variance (noise)

\[
\langle \delta q^3 \rangle = (q - \bar{q})^3
\]
- skewness (asymmetry of the distribution)

\[
\langle \delta q^4 \rangle = \bar{q}^4 - 3(\bar{q}^2)^2
\]
\[
\delta q = q - \bar{q}
\]
**FCS - examples**

- **Poissonian**  \[ p_q = e^{-\bar{q}} \frac{\bar{q}^k}{k!} \]
  \[ k \geq 0 \]
  \[ \chi_P = \exp \left\{ (e^{i\lambda} - 1)\bar{q} \right\} = e^{\bar{q}(u-1)} \]
  \[ \langle \langle \delta q^k \rangle \rangle = \bar{q} \]

- **Binomial**  \[ p_q = C_N^k p^k (1 - p)^N - k \]
  \[ k \geq 0 \]
  \[ \chi_B = (pu + 1 - p)^N \]
  \[ \bar{q} = Np \]
  \[ \langle \langle \delta q^2 \rangle \rangle = Np(1 - p) \]

- **Bidirectional Poissonian**  \[ \chi_{2P} = e^{\bar{q}(u-1)} e^{\bar{q}(u^{-1} - 1)} \]
  \[ u = e^{i\lambda} \]
Main result

- Assumptions: \( T=0 \), adiabatic pumping, no interactions but any number of channels and any evolution of scatterer.

- Zeros of \( \chi(u) \) considered as a function of complex variable \( u = e^{i\lambda} \) are real and negative.

- Zeros are given by the spectrum of \( \mathcal{F} \rightarrow \mathcal{F} \).
Implications: forbidden statistics (no-go theorem)

Not all statistics can be realized!

Example: \[ p_{0,\pm 1} = 1/3 \]

\[ \chi(u) = \frac{1}{3} (u^{-1} + 1 + u) = 0 \]

\[ u = e^{\pm i \frac{2\pi}{3}} \quad \text{Not allowed!} \]
Adiabatic Pumping  
(instant scattering)

The point contact is described by an instantaneous S-matrix: $S(t, E)$ changes slowly both as a function of energy and time (adiabaticity condition)

$$\hbar \frac{\partial S^\dagger}{\partial t} \frac{\partial S}{\partial E} \ll 1$$

We assume energy-independent scattering

$$S(t) \in U(2M)$$  \( M \) - number of channels in each lead

in-out representation

$$\begin{pmatrix} R_{out} \\ L_{out} \end{pmatrix} = S \begin{pmatrix} R_{in} \\ L_{in} \end{pmatrix}$$
Scattering matrix

single channel $M=1$: $S \in U(2)$

$$S(t) = e^{i \frac{\varphi}{2}} \begin{pmatrix} \sqrt{1 - ge^{i\alpha}} & i \sqrt{ge^{i\beta}} \\ i \sqrt{ge^{-i\beta}} & \sqrt{1 - ge^{-i\alpha}} \end{pmatrix}$$

conductance $G$ and three phases $\alpha, \beta, \varphi$ are functions of time

bias voltage $V(t)$: $\beta \rightarrow -\phi(t) = \frac{e}{\hbar} \int_{-\infty}^{t} V(t) \, dt$

$R \rightarrow e^{i\phi/2} R; \quad L \rightarrow e^{-i\phi/2} L$
FCS - general formalism

\[ \chi(\lambda) = \text{Tr} \left( \rho_0 U^\dagger e^{i\lambda P_3} U e^{-i\lambda P_3} \right) = \left\langle e^{i\lambda(Q-Q_0)} \right\rangle \]

Tr is taken over the Fock space

\[ \rho_0 = \frac{1}{Z} e^{-\beta H_0} \]
- initial state density matrix

\[ U \]
- evolution operator from initial time to the time of measurement

\[ P_3 \]
- operator of the charge of the right lead

all operators - exponents of fermionic bilinears!
Levitov-Lesovik formula

\[ \chi(\lambda) = \text{Tr} \left( \rho_0 U^\dagger e^{i\lambda P_3} U e^{-i\lambda P_3} \right) \]

\[ \chi(\lambda) = \det \left( 1 - n_F + n_F S^\dagger e^{i\lambda P_3} S e^{-i\lambda P_3} \right) \]

\[ n_F = \frac{e^{-\beta H_0}}{1 + e^{-\beta H_0}} \quad \text{fermionic occupation operator} \]

\[ S(t) \quad \text{time-dependent U(2) S-matrix} \]

\[ P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{projector on right lead} \]
z-representation (T=0)

\[ S^\dagger = \begin{pmatrix} z_1 & \tilde{z}_1 \\ z_2 & \tilde{z}_2 \end{pmatrix} = (z|\tilde{z}) \]

unitarity: \[ z^\dagger z = \tilde{z}^\dagger \tilde{z} = 1 \]
\[ \tilde{z}^\dagger \tilde{z} = 1 \]

\[ \chi(\lambda) = \det \left( e^{-i\lambda n_F} z^\dagger e^{i\lambda n_F} \tilde{z} \right) \]

FCS depends on only on \( z(t) \) - independence on \( L_{out} \) states

Gauge transformation: \( z(t) \rightarrow z(t) e^{i\gamma(t)} \)

changes only the total pumped charge: \( \chi(\lambda) \rightarrow \chi(\lambda) e^{-i\lambda \int \frac{dt}{2\pi} \dot{\gamma}} \)

FCS depends only on the unit vector:
\[ \vec{N}(t) = z^\dagger \vec{\sigma} z \]
“Bloch” sphere

Transferred charge is a geometric invariant of the path

\[ \bar{q} = \frac{A_{sphere}}{4\pi} \quad \text{Brouwer, 1998} \]

Noise is a geometric invariant of some surface enclosed by the path

\[ N_z = 1 - 2g \quad \text{Makhlin, Mirlin, 2001} \]

Minimized noise - the area of a soap film bounded by the path

\[ \langle \langle \delta q^2 \rangle \rangle_{min} = \frac{A_{min}}{4\pi} \]
$n^z_F$ - operator

$$\chi(\lambda) = \det \left( e^{-i\lambda n_F} z^\dagger e^{i\lambda n_F} z \right)$$

$$\chi(u) = e^{i\lambda \text{Tr}(n_F - n^z_F)} \det \left( e^{-i\lambda n^z_F} \left[ 1 + (u - 1)n^z_F \right] \right)$$

FCS is fully (up to the total transferred charge) determined by the spectrum of $n^z_F$.

$$\bar{q} = \text{Tr} \left( n_F - n^z_F \right)$$

$$n^z_F = z^\dagger n_F z$$

FCS zeros are determined by the spectrum of $n^z_F$!
Zeros of FCS

\[ \chi(u) = 0 \quad \Rightarrow \quad 1 + (u - 1) n_{F}^{\dagger} = 0 \]

\[ u = -\frac{1 - n_{F}^{\dagger}}{n_{F}^{\dagger}} \quad \Rightarrow \quad n_{F} = 0, 1 \quad -\text{projector} \]

\[ n_{F}^{\dagger} = z^{\dagger} n_{F} z \quad z^{\dagger} z = 1 \]

\[ n_{F}^{\dagger} \in (0, 1) \]

Complex conjugated roots are not allowed.

Spectrum of \( n_{F}^{\dagger} \) gives probabilities of independent tunneling events

\( FCS \) zeros are real and negative!
Bias voltage - “Bloch” sphere

\[ W = \frac{e}{2\pi\hbar} \int_0^T V(t) \, dt \]

\[ \bar{q} = \frac{\Delta \phi (1 - N_z)}{4\pi} = gW \]

For optimal voltage pulses

\[ \langle \langle \delta q^2 \rangle \rangle_{\text{min}} = \frac{A_{\text{min}}}{4\pi} = g(1 - g)W \]

\[ \chi(u) = (1 - g + gu)^W \]

\[ N_z = 1 - 2g \]
Bias voltage case

Time-independent scatterer with applied bias voltage

\[ g = \text{const} \]
\[ \phi(t) = \frac{e}{\hbar} \int_{t}^{t} V(t) \, dt \]
\[ n_F^z = (1 - g)n_F + gn_F^\phi \]
\[ n_F^\phi = e^{-i\phi} n_F e^{i\phi} \]
\[ z = \begin{pmatrix} e^{i\phi(t)} \sqrt{1 - g} \\ \sqrt{g} \end{pmatrix} \]
\[ N_z = 1 - 2g \]
\[ N_x + iN_y = 2\sqrt{g(1 - g)} e^{i\phi(t)} \]

Spectral properties of \( n_F^z \) in this case?

Additional restrictions on FCS zeros?
Bias voltage case

Additional restrictions on FCS zeros? Vanevic, Nazarov, Belzig, 2007

\[ n^\dagger_F = (1 - g)n_F + gn^\phi_F \]

- typical eigenvalues \( n_F \leftrightarrow 1 - n_F \)
- special eigenvalues \( n^\dagger_F = -g/(1 - g) \)

\[ W \quad \text{degeneracy} \]
\[ W = \frac{e}{2\pi\hbar} \int_0^T V(t) \, dt \]

\[ \chi(u) = \chi_0(u)\chi_{inv}(u) \]
\[ \chi_0(u) = [1 - g + gu]^W \]
\[ \chi_{inv}(u) = \det \left[ 1 + g(1 - g)(u + u^{-1} - 2)(1 - n_F)n_F^\phi \right] \]
\[ \chi_{inv}(u) = \chi_{inv}(u^{-1}) \quad \text{If } W=0 \quad (\text{no mean voltage}) \]

\[ \bar{V} = 0 \]
\[ p_q = p_{-q} \]
Noise minimization

\[ \chi_0(u) = [1 - g + gu]^W \]

\[ \chi_{\text{inv}}(u) = \det \left[ 1 + g(1 - g)(u + u^{-1} - 2)(1 - n_F)n_F^\phi \right] \]

\[ \langle \langle \delta q^2 \rangle \rangle = \text{Tr} \left[ (1 - n_F)n_F^\phi \right] \]

\[ e^{i\phi(z)} = \prod_{i=1}^{W} \frac{z - a_i}{1 - a_i^* z} \quad |a_i| < 1 \quad z = e^{i\omega t} \]

Minimize noise by choosing the shape of \( V(t) \)

\( V(t) \) - superposition of \( W \) Lorentzians of unit area

\[ \chi(u) = [1 - g + gu]^W \]

Binomial statistics
Allowed statistics

\[ \chi(u) = u^{N_1} \prod_{i=1}^{N_2} \frac{u - u_i}{1 - u_i} \]

One can approximate given FCS by sharp separated Lorentzian pulses

\[ \chi(1) = 1 \]
Many channels

All results can be generalized to $M$ channels

$z$ - $2M \times M$ matrix

$N \in \frac{U(2M)}{U(M) \times U(M)}$
Conclusions

• Constraints on the charge-transfer statistics are derived

• Zeros of FCS generating function are real and negative
  • $T=0$
  • no interactions
  • instant (energy-independent) scattering approximation

• Forbidden statistics exist

• Connection to bias-voltage case
• $n_F^Z$ operator, fully describing quantum noise is introduced

Trace identities

\[ \Gamma(A) = \sum_{i,j} c_i^\dagger A_{ij} c_j \]

\[ \text{Tr} \left( e^{\Gamma(A)} \right) = \det (1 + e^A) \]

\[ Z = \text{Tr} \left( e^{-\beta \Gamma(H_0)} \right) = \det (1 + e^{-\beta H_0}) \]

\[ \text{Tr} (1) = \det (1 + 1) = 2^N \]

\[ \text{example: partition function of free fermions} \]

\[ \text{example: number of states} \]

\[ \text{generalization} \]

\[ \text{follows from: } \quad e^{\Gamma(A)} e^{\Gamma(B)} = e^{\Gamma(C)} \]
Levitov’s formula

\[ \chi(\lambda) = \det \left( 1 - n_F + n_F S^\dagger e^{i\lambda P_3} S e^{-i\lambda P_3} \right) \]

Periodic evolution of $S(t)$ - discrete energy levels
Matrices in energy space and in channel space

T=0
\[ n_F(E) = \theta(-E) \]
\[ n_F(t, t') = \frac{i}{2\pi (t - t' + i0)} \]
\[ n_F^2 = n_F \quad \text{projector to negative energies} \]

at high positive energies matrix tends to 1
at high negative energies even at constant $S$ \[ S^\dagger e^{i\lambda P_3} S e^{-i\lambda P_3} \]

Regularization is needed!
Regularization

\[ \chi(\lambda) = \det (1 - n_F + n_F S^\dagger e^{i\lambda P_3} S e^{-i\lambda P_3}) \]

\[ \chi(\lambda) = \det \left\{ e^{-i\lambda P_3 n_F} \left[ 1 - P_3 + P_3 S e^{i\lambda n_F} S^\dagger \right] \right\} \]

Avron, Bachmann, Graf, Klich, 2007
Muzykantskii, Adamov, 2003

for finite matrices - same as Levitov’s, but well-defined

\[ e^{i\lambda n_F} = 1 - n_F + n_F e^{i\lambda} \]
\[ e^{i\lambda P_3} = 1 - P_3 + P_3 e^{i\lambda} \]

projector properties
useful for derivations