Critical phenomena and renormalization group: a simple historical overview

Carlo Di Castro
Dipartimento di Fisica Università di Roma “La Sapienza”
Collective properties of many body systems arise as a consequence of the interaction among the “elementary objects” (atoms, electrons, spins…). This reductive scheme has been at the basis of the statistical mechanics construction.

Then macroscopic physical systems show a variety of behaviors and different properties by varying the external parameters like temperature, pressure, magnetic field … and in particular appear in different form of aggregation (solid, liquid, gas; paramagnetic, ferromagnetic; superconductors, normal metals; super and normal fluid…).
E.G. Liquid Helium and the lambda transition.
Complexity

It is now fashionable to call science of complexity all that refers to those systems for which the knowledge of the individual constituents does not provide directly the knowledge of the system as a whole.

The system is said to acquire “emergent” properties in the philosophical sense of non preexisting properties. Actually the properties of a macroscopic system are evaluated as Response functions to external actions: they can be simple (as the pressure of a gas on the walls of a container or the Drude conductivity of a simple metal) or difficult (complex?) to calculate or even to individuate (like in spin glass or in the metal-insulator transition).

To refer generically to complexity and emergent properties may exalt for the non experts (with a non careful popularization) the magic aspect of physics at the expenses of its rational and predictive power producing a crises of science itself.
Reductivism and complexity

For all the properties or response functions evaluated starting from the elementary objects up to the nineties of the last century never has been advocated the word “emergent” even when the realization and the evaluation of the appropriate response were difficult (complex?) or as in the case of critical phenomena the response of the macroscopic system cannot be simply obtained as a sum of microscopic events.

If the complexity is referred to the properties that develop by considering a big number of the simple objects constituting the system, are these new aspects (as usually stated) in contrast with the reductivism and the predictive power of physics, in particular of condensed matter?

I will try to answer this question by showing how the reductivism is at work in critical phenomena often indicated as part of “complexity” and as “the” example of inapplicability of the reductionism scheme.
Naive reductivism

Of course one should not refer to the naïve reductionism of the first decades of the last century.

E.G. : J. Jeans “The mysterious universe” (1933):

“It [carbon atom as constitutive of macromolecules] appears to differ from its nearest neighbours in the table of chemical elements (B, N) only in having one electron more than the former and one electron fewer than the latter. Yet this slight difference must account in the last resort for all the difference between life and absence of life.”

“Magnetism depends on the peculiar properties of 26 [Iron], 27 [Nickel] and 29 [Cobalt] electron atoms, especially the first”
Many body systems

Prevalent procedure to obtain the behavior of macroscopic systems and thermodynamics:
N “particles” ⇒ equations of motion and dynamics ⇒ average according to the Gibbs prescription⇒ macroscopic description in terms of few thermodynamic variables T, ρ, p; h, m …

Loss of information considered inessential to the problem under consideration with a contraction of number of variables and a simplified description of the problem.

Great number of particles forbids the direct solution starting from the elementary objects. It becomes the simplifying element for the new scheme by allowing the average procedure.
Many body systems 2

Technical improvement in the first half of the 20th century required a better knowledge of the variety of the condensed matter systems⇒ simplification at work mainly in the fifties up to the seventies:

At the **phenomenological level** the main idea was the concept of **quasiparticles** as elementary excitations of the system: a system of strongly interacting particles (electrons in metals, superfluid Helium, superconductors, magnets…) **was considered at sufficiently low temperature** as a gas of quasiparticles plus at most a **weak residual interaction** (normal Fermi liquid, phonons and rotons, gapped excitations, spin waves…).
Example: Normal Fermi Liquid

- Free Fermions
- Fermi Surface
- Coherent motion (Bloch theory)

In metals electrons interact strongly

However they screen each other and at low energy the elementary excitations (QP) are like free fermions with some dressed parameters \( (m_0 \rightarrow m^*) \)

Analogously for specific heat \( C_v \) and susceptibility \( \chi \)

Gas of Fermions: Trivial Statistics
In the **many body theory** it was then privileged the development of techniques to obtain from the microscopic description for each specific system this simplified dynamics. The statistical part was then trivial, a gas with at most a normal distribution of fluctuations vanishing in the thermodynamic limit (1/N^{1/2} law).

The entire word of condensed matter was considered to be reducible to a collection of quasiparticles whose effects were obtainable in perturbation theory, even though a sophisticated one.
Crises of previous reduction scheme

In the late sixties of the last century it became clear in the study of critical phenomena that the previous scheme was inadequate. The collective phenomena does not arise as a simple superposition of single microscopic events. Through the ordering action of forces the laws of great numbers are modified (violation of $1/N^{1/2}$ law). It is no more true that each portions, sufficiently large, has an average behavior independent from the rest. The statistical aspect is now dominating showing a universal character. Up to the sixties each specific system had been studied for its specific property like for instance superfluidity in Helium. Later on all of them started to be considered as an unicum as far as their changes of states are concerned. Landau already in 1937 unified mean field theory of critical phenomena via the concept of order parameter, but it took other 30 years to rich an understanding of the problem.
Critical Phenomena

Critical point

\[ T = T_c \]

\[ T < T_c \]

\[ T = T_c \]

 Broken symmetry \( m_0(T) \approx |T - T_c|^{\beta} \)

\[ \chi_T^{-1} = (\partial m / \partial h)^{-1}_{H=0} \approx |T - T_c|^{\gamma} \]

\[ \kappa_T^{-1} = -v \left( \frac{\partial P}{\partial v} \right)_T \]

\( \chi_T \to \infty \), Strong order parameter fluctuations near the instability
Landau theory and Gaussian fluctuation

The prototype model of an interacting system at a temperature $T$ is the $\phi^4$ model, described by the effective Hamiltonian in $d$ spatial dimensions

$$\frac{1}{T}H_{\text{eff}}[\phi] = \int [(\nabla \phi)^2 + t\phi^2 + u\phi^4]d^d r,$$

Where the order parameter $\phi$ is the thermal average of a suitable classical field $\hat{\phi}$ and $t \approx T-T_c$ which acts as a “mass”

The partition function $Z = \int D\phi e^{-H_{\text{eff}}[\phi]}/T$

$F_L = H_{\text{eff}}[\phi]$ with $\delta H_{\text{eff}}/\delta \phi = 0$ self consistence equation, no fluctuations $\phi^2 = -t/2u$ for $T<T_c$ and $\phi^2 = 0$ for $T<T_c$

Gaussian fluctuations $(\Gamma^{(2)})^{-1} = \langle \delta \phi_k^2 \rangle = 1/(\xi^{-2} + k^2)$ and $\xi^{-2} = \chi^{-1} = 2t + 12u\phi^2 \approx |t|$, $x_0^2 = 2$ classical dimension of $t$ in (length)$^{-1}$

Correction due to Gaussian fluctuations are no relevant as long as the dimensionless coupling $u/t^{(4-d)/2} < 1$ Ginzburg criterion $\xi^{d+2} < |t|/u$

d<4 $u_t \Rightarrow \infty$ no perturbation theory, infrared singularities
Infinite correlation length

Near the instability strong fluctuations with bubbles of size $\xi$ of the wrong phase in the dominating phase $\xi \to \infty$ as $T \to T_c$ $kT_c = \Delta U \xi^d$

A great (infinite) number of degrees of freedom are strongly correlated and $\xi \to \infty$ invalidates the previous interpretative schemes but at the same time together with universal behavior and power laws provides in the sixties the via to the new scheme. L.P. Kadanoff; A. Z. Patashinkij, V.L. Pokrovskij; Widom….

Two forms of universality:
1) The details specifying the system do not matter
2) The degrees of freedom at short distance do not influence the critical behavior
Universality 1

If the details specifying the system do not matter, the great (infinite) number of degrees of freedom involved are well accounted for by a small set of parameters. Once the proper choice of the relevant parameters, e.g. \( t \) and \( \varphi \) (or the conjugate field \( h \)) is made, we can change the parameters \( \{\lambda_i\} \) specifying the details of the system while maintaining the physics at criticality provided we rescale the relevant ones

\[
F[t, \varphi, \{\lambda_i\}] = F[t', \varphi', \{\lambda_i'\}]; \\
\varphi[t, h, \{\lambda_i\}] = a[\{\lambda_i\}, \{\lambda_i'\}]\varphi[t', h', \{\lambda_i'\}]
\]

\[\ldots\]

Too general to be meaningful.

\( \xi \to \infty \) and power laws
Universality 2

ξ→∞ the scale of length is one of this irrelevant variables.
The original system and a block variables system should have the same critical behavior

\[ r_0 \Rightarrow r_0' = sr_0 \quad \Lambda' = \Lambda/s = 2\pi/sr_0 \]

Power laws

\[ \varphi(r_i, t, h) = s^{-x_\varphi} \varphi(r_i/s, t', h') \]

\[ t' = s^{x_t} t \quad h' = s^{x_h} h \]

Physics independent from s.

Homogeneity relations and 7 relations among the 9 critical indices

Relevant (scaling dimension x>0) and irrelevant variables (x<0)
Field Theory Renormalization Group (RG)

At the end of the sixties the phenomenological theory of critical phenomena had found sound basis in the logical sequence of universality, scaling, relevant and irrelevant variables. In 1969 the FTRG approach was introduced in critical phenomena simultaneously in Russia (Migdal, Polyakov, Larkin and Khmel’nitskii) and in Rome (CDC and Jona-Lasinio).

We realized that the FTRG (Gell-Man and Low, Bogolubov and Shirkov, Bonch-Bruevich and Tyablikov) generalizes homogeneity relations and implements universality: RG relates one model system to another by varying the coupling $u$ and suitably rescaling the other variables and the correlation functions (vertices and propagators): in ordinary critical phenomena described by the $\phi^4$ model the diagrammatic structure is invariant by renormalizing the coupling and rescaling the field $\phi$ (wave function renormalization $z_\phi$), $t$ (mass renormalization $z_t$) and the four point vertex $\Gamma_4$ (with $z_u$) to take care of the singularities in P.T.

When the coupling does not change anymore (fixed point) scaling and expressions of critical indices follow.
\[
\Gamma_0^{(2)} = |k|^2 + t_0; \\
\Gamma^{(2)} = \Gamma_0^{(2)} + \text{Diagram 1} + \text{Diagram 2}
\]

**FIGURE 1.** Dyson equation for the inverse propagator \( \Gamma^{(2)} \).

\[
\Gamma^{(4)} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \ldots
\]

**FIGURE 2.** Diagrams contributing to the dressed vertex \( \Gamma^{(4)} \).

\[
\gamma_t = \frac{\partial \Gamma^{(2)}}{\partial t}.
\]

\[
\gamma_t = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \ldots
\]

**FIGURE 3.** Diagrammatic expansion for \( \gamma_t \) (full dot). Each empty dot represents the mass insertion due to the derivative of \( \Gamma^{(2)} \) (see Fig. 1) with respect to \( t \).
The diagrammatic structure, diagram by diagram, and therefore the full quantities are invariant:

Under a wave-function renormalization

\[ \hat{\phi} \rightarrow Z_{\phi}^{-1/2}\hat{\phi}, \quad u \rightarrow Z_{\phi}^2 u, \quad \gamma_{\phi} \rightarrow Z_{\phi}\gamma_{\phi}; \]

Under a mass renormalization

\[ t \rightarrow Z_{t}^{-1}t, \quad \gamma_{t} \rightarrow Z_{t}\gamma_{t}; \]

Under a vertex renormalization

\[ u \rightarrow Z_{u}^{-1}u, \quad \gamma_{u} \rightarrow Z_{u}\gamma_{u}; \]

with

\[ \gamma_{\phi} = \frac{\partial \Gamma^{(2)}}{\partial |k|^2}, \quad \gamma_{t} = \frac{\partial \Gamma^{(2)}}{\partial t}, \quad \gamma_{u} = \frac{\Gamma^{(4)}}{u}, \]
Considering the three transformations simultaneously:

\[ \hat{\phi}' = Z_\phi^{-1/2} \phi, \quad t' = Z_\phi Z_t^{-1} t, \quad u' = Z_\phi^2 Z_u^{-1} u \]

we have the **invariant thermodynamic potential** and a multiplicative transformation of correlation function and vertices

\[ \bar{\Gamma}(\hat{\phi}', t', u') = \Gamma(Z_\phi^{1/2} \phi', Z_\phi^{-1} Z_t t', Z_\phi^{-2} Z_u u') \]

where \( i = \varphi, t, u \)

The normalization condition permits to express the \( \{ z_i \} \) in terms of the \( \{ \gamma_i \} \) as a function of \( \lambda \) and one of the variables \( \varphi, t \) or \( u \).

We can therefore change \( \phi', t', u' \) by varying \( \lambda' = \lambda/s \)

The \( \gamma_i \) can be normalized to one at a given point (n.p.) specified by an auxiliary variable \( \lambda \) with dimension of (length)\(^{-1}\)

\[ \bar{\gamma}_{\varphi}|_{\text{n.p.}} = 1, \quad \bar{\gamma}_t|_{\text{n.p.}} = 1, \quad \bar{\gamma}_u|_{\text{n.p.}} = 1 \]
For example: since we expect \( \phi' \) and \( t' \) to be the relevant variables, it is natural to parameterize the transformation in terms of \( u' \) only by the choice

\[
\text{n.p.} = (k^2=0, \phi' =0, t' = \lambda^2; u')
\]

In this way the transformation is multiplicative and linear in the variables \( \phi' \) and \( t' \).

This procedure implements the physical idea that, when the first irrelevant coupling \( u \) is changed (and asymptotically reaches a finite value \( u^* \) where does not change anymore), a proper rescaling of the relevant variables yields a system which shares the same critical properties of the initial system (with critical indices given in terms of \( u^* \) and not on \( u \)).

Different choices of the n.p. lead to different realization of RG asymptotically equivalent in the infrared region and giving the same expressions for the critical indices.
Block variables

In 1971 Wilson gave his great contribution to the understanding of the physics underlying scaling. His RG implements the second idea of universality of the elimination of degrees of freedom at short distances ($r_0 < r < r_0 s$).

\[
\phi_j(s) = s^{-d} \sum_{i \in j} \phi_i \quad \phi_{j/s} = s^{x_\phi} \phi_j
\]

To have a well defined asymptotic probability distribution when $s \to \infty$. 

\[ x_\phi = (d-2+\eta)/2 \]

With normal distribution (with $1/(N)^{1/2}$ law) \( x_\phi = d/2 \)

\[
P[\{\phi_i\}] = \frac{1}{Z} e^{-H[\{\phi_i\}]}, \quad \left\langle \phi_N^2 \right\rangle = N \sigma^2
\]

\[
P_s[\{\phi_{i/s}\}] = \int \left[ \prod_i d\phi_i \right] P[\{\phi_i\}] \prod_j \delta \left( \phi_{j/s} - \frac{s^{x_\phi}}{s^d} \sum_{i \in C_j} \phi_i \right)
\]
The Wilson transformation in momentum space

By integrating out the short wave length fluctuations of the field $\tilde{\varphi}$ with momenta $\Lambda/s<k<\Lambda$, the Wilson transformation reads

$$e^{-H_s-F_s^{(0)}} = \left( \int \left[ \prod_{\Lambda/s<|k|<\Lambda} d\hat{\varphi}_k \right] e^{-H[\hat{\varphi}]} \right)_{q\to q/s, \hat{\varphi}_q \to s^{1-n/2}\hat{\varphi}_s q}$$

When $s\to\infty$, on the critical surface the transformation reaches a well defined non Gaussian distribution with a fixed point Hamiltonian $H^\ast$. Non classical critical indices were calculated for the first time either by numerical evaluation of the relevant equations or by the $\varepsilon$-expansion (Wilson and Fisher, Wilson, 1972). Considering $\varepsilon=4-d$ a small parameter, perturbative terms of the transformation are evaluated starting from the known case at $d=4$. The same results were obtained with FTRG (CDC,’72; Brezin et al’73;,,,).
General RG Transformation

Many realizations of RG have been proposed [(Brezin et al 1973),
Bell-Wilson, 1975,…]
Some are multiplicative on the variables and correlation functions.
Some change number of variables and mix correlation functions.
All of them must be asymptotically equivalent.(Jona-Lasinio, Nobel
In general, given a space $\mathbf{H}$ of Hamiltonian $H$ (specified by a set of
parameters $\{\mu_i\}$) and a RGT $R_s$ such that

$$R_s(H) = H' \in \mathbf{H} \quad \text{or} \quad \mu_i' = f_i^s(\{\mu_j\}),$$

the invariance property of the thermodynamic functional

$$F(R_s(H)) = F(H) \Rightarrow d_RF = 0$$

provides (by functional derivates) all RG equations for the various
quantities.
General properties of RGT

-Composition law: \( R_s(R_{s'}(H)) = R_{ss'}(H) \)

-The transformation must be regular and not introduce singularities:
Given a small perturbation \( \tau \delta H \ (\tau \ll 1) \) to \( H \) and the linearized RGT \( L_s \)

\[
R_s(H + \tau \delta H) = R_s(H) + \tau L_s(H)\delta H
\]

\[
\delta \mu'_i = \sum_j \frac{\partial f_{s,i}}{\partial \mu_j} \delta \mu_j
\]

-At criticality, \( \xi = \infty \), iterating the RGT a non trivial fixed point must be reached: \( R_s(H^*) = H^* \) such that \( R_s(H_c) \Rightarrow H^* \) as \( s \Rightarrow \infty \). ∴

\[
R_s(R_{s'}(H + \tau \delta H)) = R_{ss'}(H + \tau \delta H) = R_{ss'}(H) + \tau L_{ss'}(H)\delta H.
\]

\[
R_s(R_{s'}(H + \tau \delta H)) = R_s(R_{s'}(H) + \tau L_{s'}(H)\delta H)

= R_{ss'}(H) + \tau L_s(R_{s'}(H))L_{s'}(H)\delta H,
\]
Linearized RGT eigenvectors and eigenvalues

The eigenvectors of the linearized RGT with negative scaling dimensions \((x_l < 0)\) define the tangent plane to critical surface and correspond to the irrelevant variables. The eigenvectors with positive scaling dimensions \((x_l > 0)\) define the directions of escape from critical surface and are the relevant variables. Universality classes as domains of attraction of different fixed points.

Standard renormalization: a finite (small) set of renormalized parameters to take care of the singularities in perturbation theory. Relevant and renormalized parameters coincide asymptotically in the infrared region. Each new renormalization brings in a new critical index → variety of critical phenomena (anisotropy, …)
Multiplicative structure associated to a general RG

Generalize the previous assumption of the existence of the set \( \{ h_1^* \} \)
And assume a set \( \{ h_l(H) \} \) such that
\[
L_s(H)h_l(H) = \lambda_l(H,s)h_l(R_s(H)) \quad \text{Lim}_{s \to \infty} h_l(R_s(H)) = h_1^*
\]
Then from the cocycle
\[
\lambda_l(R_{s'}(H),s)\lambda_l(H,s') = \lambda_l(H,ss')
\]
we obtain a multiplicative structure of the Field Theory type.
Differentiate with respect to \( s' \) and put \( s = s' \):
\[
\left[ s \frac{\partial}{\partial s} - \sum_k \psi_k \frac{\partial}{\partial \mu_k} - \sigma_i(H) \right] \lambda_i(H,s) = 0
\]
where \( \{ \mu_k \} \) set of parameters specifying \( H \);
\( \psi = dR_s(H) / ds \big|_{s' = s} \)
\( \sigma = d\lambda_i(H,s') / ds \big|_{s' = s} \)

Attractive fixed point \( \psi_k(H^*) = 0, \quad \sigma_i(H^*) = x_i \)
\[
\therefore \lim_{s \to \infty} \lambda_i(H,s) = \rho_i(H)s^{x_i}
\]
\( \therefore \) Apart from transients the scaling properties are equivalent
Sophisticated reductionism to filter the explicative paths to collective complex phenomena

The Gibbs prescription and the reductionism scheme from small and simple to the macroscopic and complex has not been put in crisis. It has been appropriately used through the elimination of the irrelevant variables and a subtle procedure of filtering those variables relevant and appropriate to the description of critical phenomena and of their self organization without solving the specific dynamic of each system. RG provides a general method to single out the dominant “paths” of formation of macroscopic “complex” behavior and of interpreting collective phenomena.
We have to be careful: under the name of “complexity” we could create a super science of “nothing” when applying these ideas in a simplicist way to a variety of problems (from biology, to economy, to politics). A specific problem is analyzed according to the general ideas given above and then it is assumed that stock market previsions, market fluctuations, or even new esthetic criteria,… appear as “emergent” properties. No choice of relevant variables is so general to allow to build a paradigm of Complexity.
For instance the collective behavior of social aggregates have a different logic with respect to physical aggregates. They have subjective and historical elements, which are absent in the interpretative models in Physics. The specific competences of each discipline have to be involved avoiding practical and cognitive easy shortcuts.